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On estimating Weibull modulus by the linear regression method

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Abstract Statistical models were developed to estimate the bias of the shape parameter of a 2-parameter Weibull distribution where the shape parameter was estimated using a linear regression. These models were formulated for 27 sample sizes from 5 to 100 and for 35 probability estimators, $P = (i - a)/(n + b)$, by varying "a" and "b". In each simulation, 20,000 trials were used. From these models, a class of unbiased estimators were developed for each sample size. The standard deviation and coefficient of variation of these estimators were compared to the bias of the estimators. The standard deviation increased while the coefficient of variation decreased with increasing bias of the shape parameter. Also, the Anderson–Darling statistics was used to determine that the normal, log-normal, 3-parameter Weibull, and 3-parameter log-Weibull distributions did not provide good fit to the estimator of the shape parameter.

Introduction

Weibull statistics is widely used to model the variability in the fracture properties of ceramics and to a lesser extent, metals. The probability, P, that a part will fracture at a given stress, σ, or below can be predicted as [1].

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$$P = 1 - \exp\left[-\left(\frac{\sigma - \sigma_T}{\sigma_0}\right)^m\right] \quad (1)$$

where σ_T is the threshold value below which no failure is expected, σ_0 is the scale parameter, and m is the Weibull modulus, alternatively referred to as the shape parameter. Equation 1 is for the 3-parameter Weibull distribution. When σ_T is taken as zero, as in ceramics, Eq. 1 reduces to a 2-parameter Weibull cumulative probability function. In the present study, only the 2-parameter case is investigated.

The shape parameter, m, in Eq. 1 has been used as a measure of reliability, and applied to brittle fracture of ceramics and mechanical properties of metals, such as tensile and fatigue results. Due to the destructive nature of testing involved in these studies, m has to be estimated from a sample, sometimes small in size, using one of the three methods: (i) linear regression, (ii) maximum likelihood, and (iii) moments. The Weibull modulus estimated by any of these methods is a statistic, referred to as $\hat{m}$, which by definition, has a distribution of its own. Moreover, the method of estimating Weibull modulus was found [1–3] to affect the average and standard deviation of the distribution of $\hat{m}$.

Although the estimation of the Weibull modulus has been investigated for more than 30 years, there remain some issues that need to be addressed, especially for the linear regression method. This study is motivated by this need and is focused on the linear regression method. The outstanding issues are first introduced and solutions are developed for each issue.

Background

If σ_T is taken as 0, as in brittle fracture, then Eq. 1 can be rearranged to obtain

$$\ln\left[-\ln\left(1-P\right)\right]=m\ \ln(\sigma)-m\ \ln(\sigma_0) \quad (2)$$

Note that the right hand side of Eq. 2 suggests a linear relationship with a slope of m and an intercept of $-m\ln(\sigma_0)$. To estimate m by using Eq. 2, a probability, P, has to be assigned to each experimental data point. There are several probability estimators available in the literature [3–12]. These probability estimators can all be written in the form

$$P=\frac{i-a}{n+b} \quad (3)$$

where i is the rank of the data point in the sample in ascending order, n represents the sample size, and a and b are numbers, such that $0 \le a \le 0.5$ and $0 \le b \le 1.0$. The common probability estimators found in the literature are provided in Table 1.

It has been shown [1–3, 13] via Monte–Carlo simulations that different probability estimators yield different levels of bias, i.e., difference between true value of the Weibull modulus, m_{true}, and the average of the estimated Weibull moduli. A common technique of determining bias is to normalize the estimated Weibull moduli by m_{true}. This estimated normalized moduli, $\hat{m}/m_{\text{true}}$, will be referred to as $\hat{m}*$. Of particular interest is the average of $\hat{m}*$, which will be referred to as M. If M is 1, then the probability estimator of $\hat{m}*$ is unbiased.

Recently there has been renewed interest in finding the combination of a and b that yields an unbiased estimate of the Weibull modulus. In one of the two recent studies, Wu et al. [13] changed a and b simultaneously to find unbiased probability estimators for sample sizes between 10 and 50 at increments of 5. Tiryakioğlu [14] held $b = 0$ and changed a systematically until 1 was included in the 95% confidence limits of the average of normalized Weibull moduli for sample sizes between 9 and 50. In other words, the mean of the generalized Weibull moduli generated from the simulation was 1. He also showed that a in Eq. 3 affects only the bias and not the standard deviation. The results of these two studies are summarized in Table 2. Note that a values for Wu et al. are generally in excess of 0.5. These results indicate that there are more than one set of solutions for the unbiased probability estimator for a given sample size.

There have been several investigations [15, 16] in which the use of correction factors was proposed to eliminate the bias of probability estimators. Gong [16] suggested that the corrected Weibull modulus, m_c, could be obtained by

$$m_c=\frac{\hat{m}}{M} \quad (14)$$

Such a correction would modify the standard deviation of $\hat{m}$, $s_{\hat{m}}$ such that

$$s_c=\frac{s_{\hat{m}}}{M} \quad (15)$$

where s_c is the standard deviation of the corrected Weibull modulus. Note that the standard deviation of $\hat{m}*$ in Eq. 15 represents the coefficient of variation of the normalized Weibull modulus. Ritter et al. [17] showed by using the law of propagation of errors that the coefficient of variation of normalized m should be a function of n such that:

Table 1 Probability estimators reported in the literature

a	b	Ref.	Equation
0.5	0	[4]	(4)
0	1	[5]	(5)
0.3	0.4	[6]	(6)
0.375	0.250	[7]	(7)
0.44	0.12	[8]	(8)
0.25	0.50	[9]	(9)
0.4	0.2	[10]	(10)
0.333	0.333	[11]	(11)
0.50	0.25	[1]	(12)
0.31	0.38	[12]	(13)

Table 2 Unbiased probability estimators reported by Wu et al. [13] and Tiryakioğlu [14] for various sample sizes

	Wu et al.		Tiryakioğlu
n	a	b	a
9			0.130
10	0.37	0.24	0.210
11			0.260
12			0.300
13			0.332
14			0.355
15	0.54	0.85	0.368
16			0.380
17			0.390
18			0.400
19			0.410
20	0.49	0.32	0.418
22			0.430
25	0.47	0.13	0.443
27			0.448
30	0.53	0.41	0.455
32			0.460
35	0.57	0.64	0.465
40	0.56	0.52	0.472
45	0.51	0.14	0.481
50	0.56	0.42	0.486

$$\frac{s_{\hat{m}*}}{M} = \frac{1}{\sqrt{n}} \quad (16)$$

which was later verified by Bergman [18] who ran Monte–Carlo simulations using Eqs. 4, 5, 6, and 8. Khalili and Kromp [1] however, plotted coefficient of variation versus $n^{-1/2}$ and noticed that the relationship proposed by Ritter et al. is valid only for $n \geq 10$. Wu et al. [13] investigated the same probability estimators as Bergman and found that coefficient of variation for the different estimators, including those that are unbiased, are approximately equal for all sample sizes, which is in agreement with previous studies [1, 3, 17, 18]. For coefficient of variation to be the same for all probability estimators that yield different averages (bias), the standard deviation and mean of estimated Weibull moduli have to be correlated. This point, however, has not been addressed in the literature.

Ritter et al. [17] ran Monte–Carlo simulations and concluded that the distribution of the estimated Weibull modulus is approximately normal. These researchers ran Monte–Carlo simulations only 100 times. It has since been shown [1, 19–21] that the distribution of m is positively skewed. Gong and Wang [19] stated that m follows a lognormal distribution for linear regression (using Eq. 4) and maximum likelihood methods. These authors used the χ^2 goodness-of-fit test for their evaluation. Barbero et al. [20] claimed that the distribution of m estimated by the maximum likelihood method is better expressed by a 3-parameter Weibull distribution. In a later publication [21], the same authors found that 3-parameter log-Weibull distribution provides a better fit to m estimated by the maximum likelihood method than lognormal and 3-parameter Weibull distribution. Recently, Tiryakioğlu [22] analyzed the distribution of m estimated by the maximum likelihood and moments methods using the Anderson–Darling goodness-of-fit test [23–25], which is much more sensitive to tails than the χ^2 test. Tiryakioğlu found that the distribution of $\hat{m}$ for $5 \leq n \leq 50$ is neither normal, lognormal, 3-parameter nor 3-parameter log-Weibull for the maximum likelihood method. For the moments method, the distribution of $\hat{m}$ was found to be lognormal for $n \geq 40$. For any sample size, the 3-parameter Weibull distribution did not provide a good fit.

The literature survey presented above indicates that these issues need to be investigated:

- How do a and b in Eq. 3 affect the average and standard deviation of $\hat{m}*$?
- Is the distribution of $\hat{m}*$for the linear regression method normal, lognormal, 3-parameter Weibull, or 3-parameter log-Weibull?

These issues have been investigated in this study and results are reported.

Research methodology

The study was conducted in two phases. In both phases, Monte–Carlo simulations were used to generate n data points from a Weibull distribution with $\sigma_0 = 1$. Since the value of m_{true} does not affect the distribution of normalized $\hat{m}$ (see Khalili and Kromp [1]), its value was kept constant at 10. In Phase 1, 27 sample sizes were used in this study, ranging from 5 to 100. For one observation, n random numbers between 0 and 1 were generated to obtain a set of σ values. Best fits using linear regression were obtained by using the 10 probability estimators presented in Table 1. For each sample size and probability estimator, the experiment was repeated 20,000 times.

In Phase 2, 24 additional probability estimators were tested, in two stages. In the first stage, 16 new probability estimators were determined by randomizing a between 0 and 0.5, and b between 0 and 1. The M values obtained by using these 16 random probability estimators were combined with the results of Phase 2 and earlier published results [14] to develop regression equations for each sample size with M as the response variable and a and b as regressors. These equations were used to determine eight more probability estimators at b = 1, 0.8, 0.7, and 0.6 such that they yielded an estimated M of 0.99 and 1.01. Therefore a total of 35 probability estimators were used to investigate the effect of a and b on M and the standard deviation of normalized Weibull modulus (s) for each sample size. As in Phase 1, for each probability estimator and sample size, the experiment was repeated 20,000 times using the same levels of σ_0 and m_{true}. The final regression equations were developed using data from all 35 probability estimators.

Results and discussion

The effect of probability estimators on M and s

The effect of sample size on M for different probability estimators is presented in Fig. 1. Note that the probability estimators have a strong effect on M. These results are in close agreement with the ones reported in earlier studies [1, 2]. The use of Eq. 4 was recommended by Khalili and Kromp because of its low level of bias. It is evident in Fig. 1b that Eq. 12 performs equally well. To the authors' knowledge, this is the first time that the performance of Eq. 12 is reported for the Weibull distribution.

Using the averages presented in Fig. 1, the results reported earlier by one of the investigators [14], and the additional 24 estimators generated by randomly choosing values for a and b, regression analyses were conducted to quantify the effect of "a" and "b" in Eq. 3 on M. Linear,

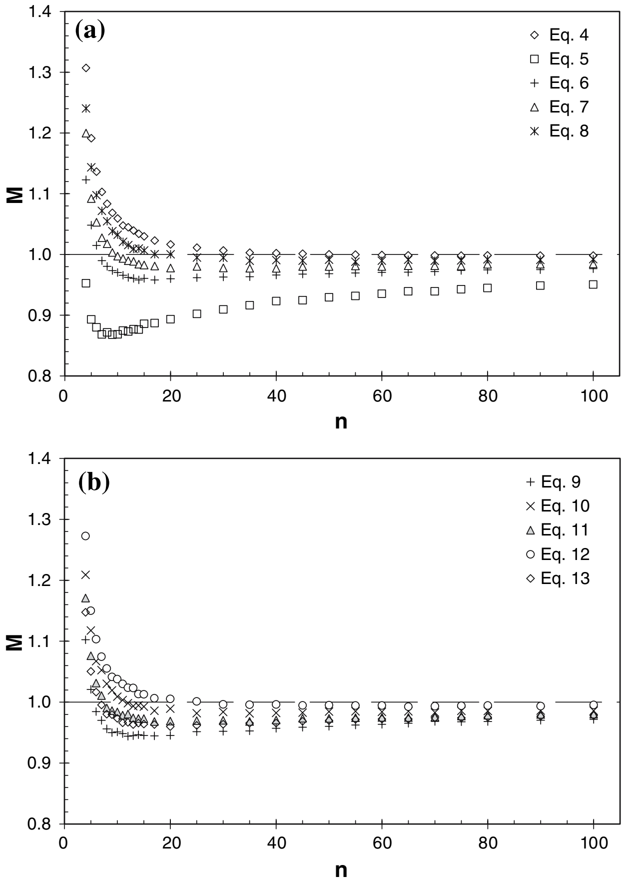

Fig. 1 The effect of sample size on M for different probability estimators found in the literature

log-linear, and quadratic models were tested. The best adjusted R^2 values (ranging from 0.991 to 0.998) were obtained in the quadratic model:

$$\hat{M} = \beta_0 + \beta_1 a + \beta_2 b + \beta_3 ab + \beta_4 a^2 + \beta_5 b^2 \quad (17)$$

where all β's are regression coefficients and their values are listed in Table 3 for all sample sizes. These regression equations can be used to estimate the bias of any probability estimator with a and b between 0 and 1.

Equation 17 along with the coefficients listed in Table 3 was used to determine the combinations of a and b that yield unbiased probability estimators, i.e., $\hat{M} = 1$. The unbiased probability estimator contours for nine sample sizes are presented in Fig 2. Note that the results of Wu et al. and Tiryakioğlu are very close to the contours for respective sample sizes. These findings show that there is more than one unbiased probability estimator for each sample size. Figure 2 also explains why Tiryakioğlu could not find levels of a that yield unbiased results for $n \leq 8$. The contours for $n \leq 8$ have no x-intercept, i.e., $b > 0$ for all values of a for the unbiased estimates.

Iso-M contours for a sample size of 20 are presented in Fig. 3. The M values for the probability estimators in Table 1 are also indicated in the figure. Note that M values for the 10 probability estimators range between approximately 0.89 and 1.02.

For each sample size, the standard deviation of normalized Weibull modulus was plotted versus M. The effect of M on the standard deviation (s) for $n = 20$ is presented in Fig. 4. Note that the standard deviation increases linearly with M. The same observation was made for all sample sizes. Therefore, standard deviation and average are correlated. To the authors' knowledge, this correlation had not been reported in previous studies.

Whether M and the coefficient of variation (s_c) are correlated was also investigated. The coefficient of variation was found to decrease with increasing M, as presented for $n = 20$ in Fig. 4. Hence the bias of a probability estimator also affects its coefficient of variation, although slightly. Again, the correlation between M and the coefficient of variation had not been reported previously. This finding indicates that correcting the Weibull modulus using Eq. 14 almost stabilizes the standard deviation. This can be seen by considering Eq. 14. The standard deviation of m_c is actually the coefficient of variation of $\hat{m}$, which changes slightly for different values of M as shown in Fig. 4. If correction factors are to be employed, it is recommended that they be applied to only Eq. 4 (and not the other probability estimators) and for $n \leq 35$ because Eq. 4 gives the highest M among the probability estimators in Table 1, as depicted in Fig. 1. For $n > 35$, all probability estimators underestimate (have negative bias), and therefore when corrected, they will have larger standard deviations than the unbiased probability estimators developed in this study. Instead, it is recommended that unbiased probability estimators found from Table 3 be used.

Since the coefficient of variation was found to be affected by M, it cannot be expected to be equal to $n^{-1/2}$ for all probability estimators. The validity of the derivation of Ritter et al. was tested by plotting $s_c n^{1/2}$ versus n for the 10 probability estimators in Table 1. The results are presented in Fig. 5. Note that for all probability estimators and sample sizes, $s_c n^{1/2} > 1$, although seems to be minimum close to unity near $n = 20$. Hence the derivation of Ritter et al. underestimates the coefficient of variation and should be used only as an approximation.

The Distribution of m

To determine whether m, estimated by the linear regression method, follows the normal, lognormal, 3-parameter Weibull or 3-parameter log-Weibull distributions, hypothesis tests were conducted using the Anderson–Darling (A^2) goodness-of-fit test statistic:

Table 3 The results of the regression analyses

n	β_0	β_1	β_2	β_3	β_4	β_5
5	1.08343	0.00650	–0.25339	0.15503	0.41598	0.06495
6	1.03645	0.05108	–0.22932	0.12063	0.30370	0.07331
7	1.01034	0.04810	–0.19986	0.11281	0.29499	0.05544
8	1.00646	–0.00189	–0.20157	0.13686	0.31602	0.06119
9	0.98350	0.07336	–0.17398	0.09494	0.19710	0.05914
10	0.99199	–0.02086	–0.17514	0.11756	0.32175	0.05006
11	0.96193	0.05300	–0.11633	0.05592	0.24680	0.02824
12	0.96290	0.04673	–0.11895	0.06600	0.22973	0.03096
13	0.95950	0.03069	–0.10180	0.05460	0.25633	0.01921
14	0.95267	0.07694	–0.10285	0.05692	0.16867	0.02590
15	0.95162	0.06955	–0.09267	0.04483	0.17340	0.02499
17	0.95003	0.06704	–0.08242	0.04516	0.15562	0.01793
20	0.94880	0.05970	–0.07162	0.03333	0.16134	0.01616
25	0.95165	0.04935	–0.06038	0.03430	0.13997	0.01069
30	0.95401	0.04162	–0.05178	0.03281	0.12813	0.00856
35	0.94962	0.05613	–0.04094	0.02240	0.10497	0.00635
40	0.95294	0.04690	–0.03760	0.01674	0.10674	0.00678
45	0.96260	0.01933	–0.03949	0.03281	0.11658	0.00211
50	0.95834	0.03969	–0.03395	0.01709	0.09412	0.00579
55	0.96050	0.04255	–0.03604	0.02479	0.07348	0.00711
60	0.96065	0.04122	–0.03250	0.02040	0.07364	0.00675
65	0.96017	0.04084	–0.02549	0.01243	0.07372	0.00480
70	0.96254	0.03306	–0.02218	0.01511	0.07559	0.00047
75	0.96214	0.04257	–0.02274	0.01559	0.05723	0.00289
80	0.96233	0.04492	–0.01939	0.00827	0.05457	0.00261
90	0.96490	0.04000	–0.01992	0.00973	0.05348	0.00373
100	0.96729	0.03368	–0.01813	0.00921	0.05659	0.00149

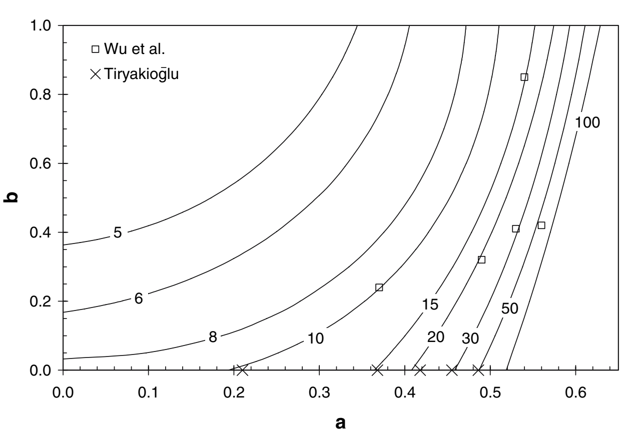

Fig. 2 The unbiased probability estimator contours for different sample sizes. The results of Wu et al. and Tiryakioğlu are also indicated

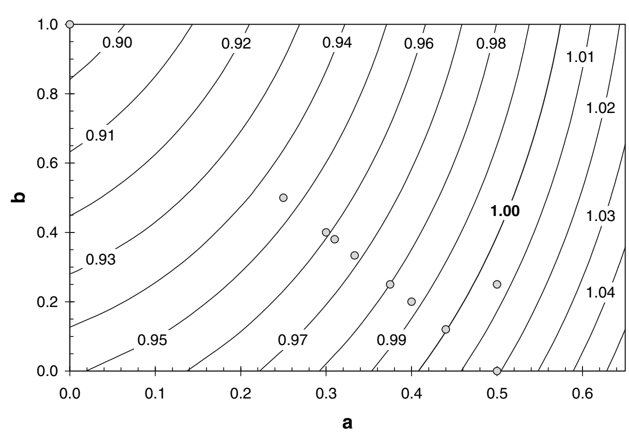

Fig. 3 Iso-M contours for n = 20. The bias of probability estimators listed in Table 1 is also indicated

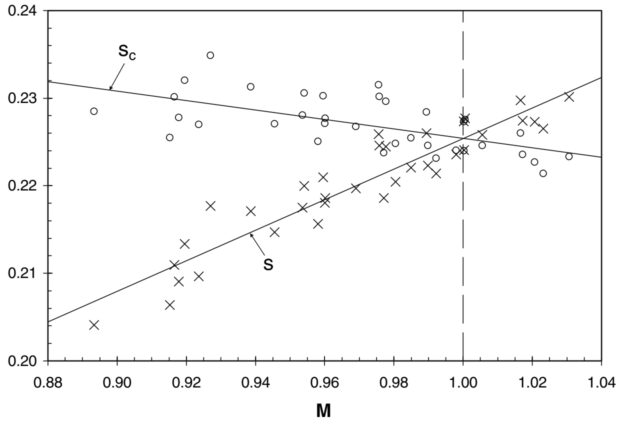

Fig. 4 The effect of M on the standard deviation of normalized Weibull modulus (s) and coefficient of variation (s_c) for n = 20

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}\left[(2i-1)(\ln P_i + \ln(1-P_{n+1-i}))\right] \quad (18)$$

The sensitivity of the Anderson–Darling test to the tails of the distribution is the reason why it was selected. Lesser the value of A^2, higher the confidence that data follow the distribution being tested.

The results of the Monte–Carlo simulations for Eq. 4 for 13 sample sizes between 5 and 100 were tested. Hence for each sample size, 20,000 data were evaluated for goodness-of-fit. The A^2 values calculated for each sample size and distribution are presented in Table 4. For each sample size and distribution, P-value is less than 0.005. Therefore m does not follow the normal, lognormal, 3-parameter Weibull, and the 3-parameter log-Weibull distributions. It should be noted that the lognormal distribution provided better fit than the other three, as indicated by significantly smaller A^2. It is not surprising that Gong and Wang found the distribution of m to be lognormal using the χ^2 goodness-of-fit test, which is much less sensitive to the tails of the distribution than A^2. Moreover m estimated by the moments method follows the lognormal distribution for

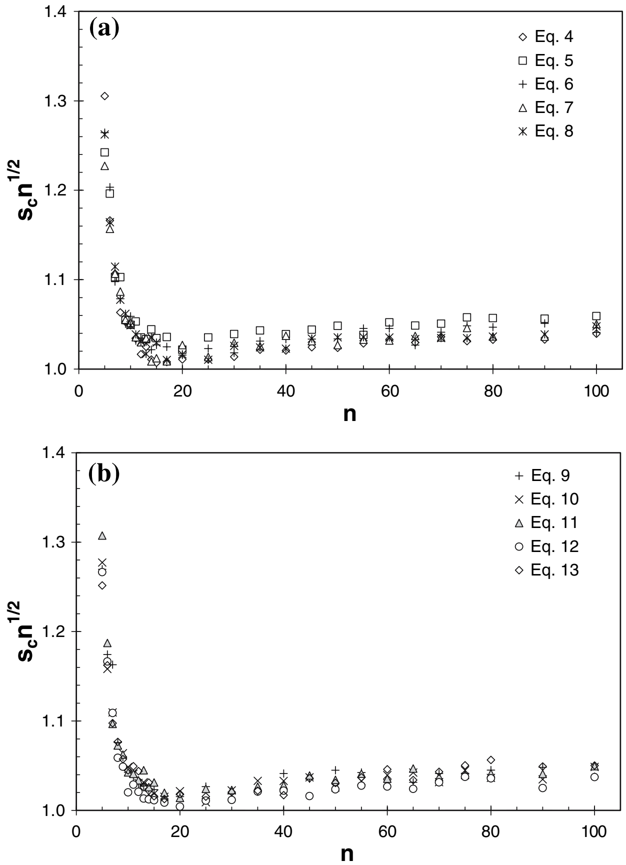

Fig. 5 The effect of sample size on $s_c n^{1/2}$ for the probability estimators in Table 1

Table 4 A^2 values for normal, lognormal, 3-parameter Weibull and 3-parameter log-Weibull to $\hat{m}/m_{true}$ data. In all cases, the P-values for the hypotheses were below 0.005

n	Normal	Lognormal	3-p Weibull	3-p Log-Weibull
5	717.11	38.03	389.67	116.64
10	220.99	4.29	194.35	62.27
15	129.41	3.83	96.93	29.04
20	83.18	2.88	102.27	35.20
25	56.03	6.01	64.96	19.92
30	45.24	6.80	72.17	26.05
40	27.36	7.00	43.17	15.84
50	25.29	3.28	88.44	43.58
60	16.33	8.74	39.10	14.00
70	15.22	7.82	59.37	27.35
80	12.34	5.46	35.81	13.58
90	10.18	5.48	53.47	26.42
100	7.80	5.86	82.03	47.62

Table 5 Average standard deviation for unbiased probability estimators of $\hat{m}/m_{\mathrm{true}}$

n	Standard deviation
5	0.5744
10	0.3321
15	0.2633
20	0.2281
25	0.2043
30	0.1870
35	0.1742
40	0.1628
50	0.1463
60	0.1338
70	0.1245
80	0.1169
90	0.1106
100	0.1046

$n \geq 40$ by using the A^2 goodness-of-fit test [22]. Nevertheless, m estimated by the linear regression method does not follow the lognormal distribution.

The standard deviation of the probability distribution for $\hat{m}/m_{\mathrm{true}}$ does, however, vary for different sample sizes. Table 5 shows the average standard deviation for all unbiased probability estimators of $\hat{m}/m_{\mathrm{true}}$ for different sample sizes. These unbiased probability estimators are those illustrated in Fig. 2. The standard deviation decreases as the sample size increases resulting in much narrower probability distributions for $\hat{m}/m_{\mathrm{true}}$ with large sample sizes.

Conclusions

- Regression equations were developed to estimate the bias of all probability estimators. These equations can be used to find unbiased probability estimators for each sample size. There is more than one unbiased probability estimator for each sample size.
- The standard deviation of $\hat{m}*$ increases linearly with the bias of the probability estimator.
- The coefficient of variation decreases slightly with increasing bias of the probability estimator.
- For $n > 35$, all probability estimators in the literature underestimate (have negative bias), and therefore when corrected, they will have larger standard deviations than the unbiased probability estimators developed in this study.
- The proposition of Ritter et al. that the coefficient of variation is equal to $n^{-1/2}$, underestimates the coefficient of variation and should be used only as an approximation.
- Weibull modulus estimated by the linear regression method does not follow the normal, lognormal, 3-parameter Weibull and the 3-parameter log-Weibull distributions.

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